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David W. Walkup

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SUMMARY

It is shown that the minimal number of interchanges necessary to transform one (0,1)-matrix into another equivalent one may be computed from the maximal number of edge-disjoint circuits in a bipartite graph derived from the difference of the matrices. This partially answers a question raised by Ryser. Two (0,1)-matrices are said to be equivalent if their difference has zero row and column sums. They are said to differ by an interchange if they are equivalent and their difference is zero except for a 2×2 minor.

1. Introduction. In this paper we obtain a partial answer in graph-theoretic form to a question raised by Ryser [2, page 68] concerning the minimal number of interchanges required to transform equivalent (0,1)-matrices into each other.

For given positive integers m and n we consider the collection of $m \times n$, (0,1)-matrices $A = \{a_{ij}\}$,

$$a_{ij} = 0 \text{ or } 1 \qquad 1 \le i \le m, \quad 1 \le j \le n .$$

We say the (0,1)-matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ are equivalent and write $A \sim B$ if and only if they have the same row and column sums, that is, if and only if

$$r_i = \Sigma_j a_{ij} = \Sigma_j b_{ij}$$

$$s_j = \Sigma_i a_{ij} = \Sigma_i b_{ij} .$$

We note immediately that $A \sim B$ if and only if $B - A \sim 0$, where 0 designates the $m \times n$ matrix of zeros.

Given a (0,1)-matrix A, we can obtain an equivalent one, A', by finding a 2×2 minor of A of the form

$$\begin{matrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{matrix} \qquad \text{or} \qquad \begin{matrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{matrix}$$

and replacing it by the other. Ryser calls this transformation from A to A' an interchange and shows [1; 2, page 68] that any matrix equivalent to A may be obtained from it by a suitable sequence of interchanges. We will show the following:

THEOREM 1. If A and B are equivalent (0,1)-matrices, then B can be obtained from A by a sequence of

$$\frac{1}{2}\alpha(A,B) - \beta(G) \tag{1.1}$$

and no fewer interchanges, where $\alpha(A,B)$ is the number of positions at which A and B disagree, G is the directed, bipartite graph derived from $B - A \sim 0$, and $\beta(G)$ is the maximum number of edge disjoint circuits in G.

Experimentation with a number of reasonably small examples has shown that determination of the maximum number of interchanges by evaluating $\beta(G)$ is considerably easier than by a direct examination of the matrices A and B. However, no simple algorithm for computing $\beta(G)$ has been found.

In §2, we develop some convenient methods and notations concerning graphs and matrices. In §3, we reprove Ryser's result that a sequence of interchanges exists, showing, in fact, that a sequence of length (1.1) exists. In §4, we prove a general result on graphs and show it implies (1.1) is a lower bound for the number of interchanges.

2. Preliminaries.

DEFINITION 1. By a _graph_ G _with multiplicities_, or _graph_ for short, we mean a set $V = \{v_1, v_2, \ldots, v_t\}$ of _vertices_ and an integer-valued function F on the ordered pairs of $V \times V$ satisfying $F(v_i, v_j) = -F(v_j, v_i)$ so that in particular $F(v_i, v_i) = 0$. We designate by $\mathcal{E}$ the collection of ordered pairs (v_i, v_j) of $V \times V$ for which $F(v_i, v_j) > 0$. We choose to write the elements of $\mathcal{E}$ in the form $E(v_i, v_j)$ and say $E(v_i, v_j)$ is an _arc of_ G _directed from_ v_i _to_ v_j _of multiplicity_ $F(v_i, v_j)$.

A graph with multiplicities may be thought of, if desired, as an undirected loopless graph where $F(v_i,v_j) \neq 0$ is a flux from v_i to v_j through the only edge connecting v_i and v_j.

The class of all graphs with given vertex set V we designate $\mathfrak{G} = \mathfrak{G}(V)$. Throughout we will suppose V is fixed but arbitrary. Of special interest is the subclass $\mathfrak{G}^* \subset \mathfrak{G}(V)$ consisting of <u>basic graphs</u> -- graphs with arcs of multiplicity 1 only. Basic graphs may be thought of as directed graphs with at most one arc, regardless of direction, connecting any distinct vertices. Given any basic graph G^* from $\mathfrak{G}^*$ we define a subset $\mathfrak{G}(G^*)$ of $\mathfrak{G}(V)$ as follows: G is in $\mathfrak{G}(G^*)$ if and only if for each arc $E(v_i,v_j)$ of G either $E(v_i,v_j)$ or $E(v_j,v_i)$ is an arc of G^*.

PROPOSITION 1. If G_1 and G_2 are graphs in $\mathfrak{G}$ with functions F_1 and F_2, then the function F given by

$$F(v_i,v_j) = F_1(v_i,v_j) + F_2(v_i,v_j)$$

is the function of a graph G which we may call the sum $G_1 + G_2$. $\mathfrak{G}(V)$ is an additive group under this composition and each $\mathfrak{G}(G^*)$ is a subgroup.

We will say a sum ΣG_i of graphs in a class $\mathfrak{G}(G^*)$ is <u>conjoint</u> if for each arc E of G^* the nonzero integers $F_i(E)$ have the same sign, that is, if there is no cancellation in forming the sum $F = \Sigma F_i$ for G, or in the undirected graph interpretation all fluxes reinforce. If, in fact, for each E in G^* at most one $F_i(E)$ is nonzero we will say the sum ΣG_i is disjoint. It will be seen that conjointness in a sum of graphs is thus a generalization of the usual concept of edge disjointness. By a <u>circuit of length</u> r (an r-<u>circuit</u>) we mean a graph in $\mathfrak{G}^*$ having exactly $r \geq 3$ distinct arcs

$$E(p_1,p_2),\ E(p_2,p_3), \ldots,\ E(p_r,p_1)$$

joining r distinct vertices $p_1, p_2, \ldots, p_r$ of V.

We say a graph is <u>conservative</u> if the sum of multiplicities of arcs leaving each vertex equals the sum of multiplicities of entering arcs. Any circuit is conservative, but also:

PROPOSITION 2. If the graph G of $\mathfrak{G}(G^*)$ is conservative, it can be written as a conjoint sum of circuits in $\mathfrak{G}(G^*)$.

If we wish to consider <u>bipartite</u> graphs we can suppose the vertex set V is the disjoint union of sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, $m + n = t$, and restrict attention to the subclass $\mathfrak{G}^\circ \subset \mathfrak{G}(V)$ containing those graphs which have no arcs connecting two points in X or two points in Y. As before we will suppose the integers m and n and the sets X and Y understood when considering a class $\mathfrak{G}^\circ$. The definitions and results on circuits, conservative graphs and subgroups $\mathfrak{G}(G^*)$ will carry over to the bipartite case.

If $\mathfrak{G}^\circ$ is the class of bipartite graphs on vertex sets X and Y of m and n elements respectively, we can define for each $m \times n$ matrix of integers $A = \{a_{ij}\}$ the graph $G(A)$ in $\mathfrak{G}^\circ$ whose function F is given by $F(x_i, y_j) = -F(y_j, x_i) = a_{ij}$. The correspondence $A \leftrightarrow G(A)$ is an isomorphism between the additive group of $m \times n$ matrices and the group $\mathfrak{G}^\circ$. Accordingly, we will speak of these matrices and graphs interchangeably when convenient.

PROPOSITION 3. An $m \times n$ matrix A is equivalent to zero if and only if $G(A)$ is conservative.

We note that if the graph $G(C)$ in $\mathfrak{G}^*$ corresponding to the matrix C is an r-circuit we may permute the rows and columns of C to obtain an $m \times n$ matrix

$$\begin{array}{|c|c|} \hline T & 0 \\ \hline 0 & 0 \\ \hline \end{array}$$

where T is an $r \times r$ matrix with r 1's on the diagonal, $r-1$ -1's on the superdiagonal and a -1 in the lower left. Propositions 2 and 3 combine to give:

PROPOSITION 4. Every $m \times n$ matrix A equivalent to zero is the conjoint sum of bipartite circuits. If the only entries of A are 0, 1 and -1, the sum is disjoint.

3. Proof that (1.1) can be attained. For any conservative graph G in some class $\mathfrak{G}(G^*)$ let $\alpha = \alpha(G)$ be the sum of multiplicities of G and let $\beta = \beta(G)$ be the largest integer for which G can be written as a conjoint sum of β circuits. It is easily seen that $\alpha(G)$ and $\beta(G)$ are independent of the particular choice of G^*. In the remainder of this section only we direct our attention exclusively to a class $\mathfrak{G}^*$ of bipartite graphs. We note that if A and B are equivalent (0,1)-matrices $\alpha(A,B)$ the number of disagreements between A and B, equals $\alpha(G(B-A))$.

LEMMA 1. If A and B are equivalent (0,1)-matrices then there exists a sequence

$$A=A_0, A_1, A_2, \ldots, A_\beta = B \qquad \beta = \beta(G(B-A)),$$

of equivalent (0,1)-matrices such that each difference

$$C_i = A_i - A_{i-1}$$

is a circuit (of length r_i) and

$$B - A = \sum_{j=1}^{\beta} C_j \tag{3.1}$$

is a disjoint sum. Moreover,

$$\alpha(A,B) = \sum_{i=1}^{\beta} r_i .$$

Proof. $B - A$ satisfies the stronger conditions of Proposition 4, hence the disjoint sum (3.1) exists. The partial sums

$$A_i = A + \sum_{j=1}^{i} C_j$$

are all equivalent to A since the C_i are equivalent to zero. The disjointness in (3.1) and the fact A and B are (0,1)-matrices imply the A_i are also (0,1)-matrices.

LEMMA 2. If A and B are equivalent (0,1)-matrices and $C = B - A$ is an r-circuit, then $r = 2s$ and there exists a sequence

$$A=A_0,\ A_1,\ A_2,\ \ldots,\ A_{s-1} = B \tag{3.2}$$

of equivalent (0,1)-matrices for which the differences

$$D_i = A_i - A_{i-1}$$

are circuits of length 4, i.e. A_i and A_{i-1} differ only by an interchange.

Proof. All graphs in $\mathfrak{G}^*$ are bipartite, hence the circuit $B - A$ has even length $r = 2s$. A weak result

$$B = A + \sum_{j=1}^{s-1} D_i'$$

for certain 4-circuits D_i' follows easily upon examination of Figure 1.

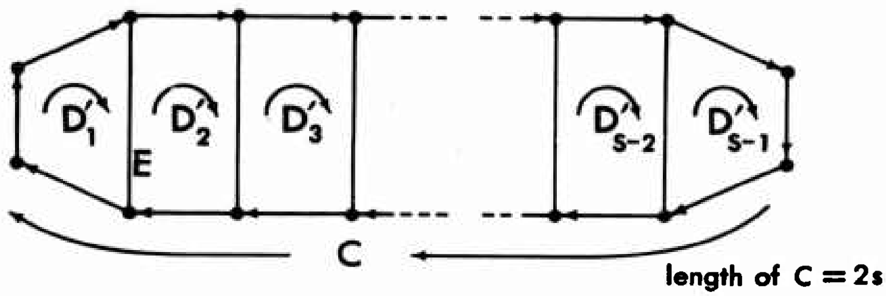

Figure 1.

Note that the D_i' will necessarily visit vertices in X and Y alternately and hence are indeed elements of $\mathfrak{S}^{\circ}$. We seek a reordering D_i of D_i' so that

$$A_i = A + \sum_{j=1}^{i} D_j$$

are (0,1)-matrices. Clearly the A_i will be equivalent. The sum

$$H = \sum_{j=2}^{s-1} D_j'$$

is a (2s-2)-circuit in $\mathfrak{S}^{\circ}$, and a matrix of zeros and ones. We assert either

(i) $\qquad A,\ A+D_1',\ A+D_1'+H = B$

or

(ii) $\qquad A,\ A+H,\ A+H+D_1' = B$

is a sequence of equivalent (0,1)-matrices. The only possible difficulty is the value of the middle terms for the ordered pair (x_i, y_j) corresponding to E in Figure 1. But D_1' and H take opposite values for this pair, hence exactly one of $A+D_1'$ and $A+H$ is a (0,1)-matrix. By applying the same argument to the circuit H instead of C we may place additional terms between $A+D_1'$ and B if (i) holds or between A and $A+H$ if (ii) holds. Repeating this process a sufficient number of times we will reach simultaneously the sequence (3.2) and the proper reordering of the D_i'.

LEMMA 3. If A and B are equivalent (0,1)-matrices there exists a sequence

$$A=A_0,\ A_1,\ A_2,\ \ldots,\ A_k = B \tag{3.3}$$

of equivalent (0,1)-matrices for which the differences $A_i - A_{i-1}$ are 4-circuits and

$$k = \frac{1}{2}\alpha(A,B) - \beta(G(B-A)).$$

Proof. The existence of the sequence (3.3) follows from lemmas 1 and 2. The value of k derives from the computation

$$\sum_{i=1}^{\beta}\left(\frac{1}{2} r_i - 1\right) = \frac{1}{2}\sum_{i=1}^{\beta} r_i - \sum_{i=1}^{\beta} 1 = \frac{1}{2}\alpha(A,B) - \beta\,.$$

4. Proof that (1.1) is a lower bound. Let G be any graph in a subgroup $\mathfrak{G}(G^*) \subset \mathfrak{G}$. We have defined $\alpha(G)$ and $\beta(G)$. For any positive integer $\delta \geq 3$ let $\gamma = \gamma(G,\delta)$ be the smallest integer for which G can be written as the sum of γ circuits from $\mathfrak{G}(G^*)$ of length δ or less. If G cannot be so written set $\gamma = \infty$.

THEOREM 2. If G is a finite, conservative graph in $\mathfrak{G}(G^*)$, then

$$\gamma(G,\delta) \geq \frac{\alpha(G) - 2\beta(G)}{\delta - 2}\quad .$$

Proof. We need consider the case $\gamma < \infty$ only. We fix δ and define the function $\phi(G)$,

$$\phi(G) = \alpha(G) - 2\beta(G) - (\delta - 2)\cdot\gamma(G).$$

We must show

$$(4.1) \qquad \phi(G) \leq 0 \qquad \text{for all } G \text{ in } \mathfrak{G}(G^*)$$

Suppose (4.1) false. Choose a graph G_0 from $\mathfrak{G}(G^*)$ for which $\alpha(G_0)$ is as small as possible subject to

$$(4.2) \qquad \phi(G_0) > 0.$$

Since the empty graph satisfies (4.1), we have

$$\alpha(G_0) > 0, \quad \beta(G_0) > 0, \quad \gamma(G_0) > 0.$$

Let

$$(4.3) \qquad G_0 = \sum_{i=1}^{\gamma} D_i$$

be some expression for G_0 as a sum of a minimum number of circuits of $\mathfrak{G}(G^*)$ of length δ or less. For each D_i let $q(D_i)$ be the number of arcs of D_i which coincide (with proper orientation) with an arc of G_0. There must exist a D_k for which $q(D_k) \geq \delta - 1$ for otherwise we would have

$$\alpha(G_0) \leq \sum_{i=1}^{\gamma} q(D_i) \leq (\delta - 2) \cdot \gamma(G_0)$$

in violation of (4.2).

We suppose first, that $q(D_k) = \delta$. Consider

$$(4.4) \qquad G' = \sum_{\substack{i=1 \\ i \neq k}}^{\gamma(G_0)} D_i.$$

By exhibiting a specific sum for G', (4.4) shows that

$$(4.5) \qquad \gamma(G_0) \geq \gamma(G') + 1.$$

Further let

$$(4.6) \qquad G' = \sum_{i=1}^{\beta(G')} C_i$$

be a representation of G' as a conjoint sum of a maximal number of circuits. Then, because $q(D_k) = \delta$,

$$G_0 = \sum_{i=1}^{\beta(G')} C_i + D_k$$

is a conjoint sum for G_0, implying

$$(4.7) \qquad \beta(G_0) \geq \beta(G') + 1.$$

Combining (4.5), (4.7) and $\alpha(G_0) = \alpha(G') + \delta$ we conclude

$$(4.8) \qquad \phi(G_0) \leq \phi(G')$$

which contradicts the choice of G_0 as a smallest graph satisfying (4.2). In the same way the assumption $q(D_k) = \delta - 1$ for a circuit D_k of length $\delta - 1$ leads to (4.8) with strict inequality.

As a third and last alternative, we assume there exists a circuit D_k in (4.3) of length δ for which $q(D_k) = \delta - 1$. Let E be the only arc of D_k which does not coincide with an arc of G_0. Again we form G' as in (4.4) and find an expansion (4.6). We check that G' is in $\mathfrak{G}(G^*)$. In G', $\delta - 1$ multiplicities of G_0 have been decreased, and one corresponding to E^-, the arc reverse to E, has been increased (possibly from zero to one). Thus

$$(4.9) \qquad \alpha(G_0) = \alpha(G') + \delta - 2.$$

As before (4.5) must hold. Let C_h be any circuit in (4.6) which has an arc coinciding with E^-. Now $C_h + D_k$ may not be a circuit, but it is a nonvacuous, conservative graph which, by Proposition 2, is the conjoint sum

$$C_h + D_k = \sum_{i=1}^{\epsilon} \tilde{C}_i$$

of at least one circuit. Therefore, we have

$$G_0 = \sum_{\substack{i=1 \\ i\neq k}}^{\beta(G')} C_i + \sum_{i=1}^{\epsilon} \tilde{C} ,$$

and this sum is easily seen to be conjoint. Accordingly,

(4.10) $\qquad \beta(G_0) \geq \beta(G')$.

But (4.5), (4.9) and (4.10) imply (4.8) again, and we are forced to conclude (4.1) always holds. This concludes the proof of Theorem 2.

Theorem 1 now follows directly from Lemma 3 and Theorem 2 for $\delta = 4$, $\mathfrak{G}(G^*) = \mathfrak{G}^\circ$.

A theorem similar to Theorem 1 can be proven for the case $\delta = 3$.

THEOREM 3. If G is a conservative graph in $\mathfrak{G}(V)$, $\alpha(G)$ is the sum of multiplicities of arcs of G, and $\beta(G)$ is the largest integer for which G can be written as a conjoint sum of circuits, then G can be written as the sum of

(4.11) $\qquad \alpha(G) - 2\beta(G)$

and no fewer 3-circuits from $\mathfrak{G}(V)$.

Proof. Theorem 2 says (4.11) is a lower bound. The proof that (4.11) can be realized follows from Figure 2 in the same way Theorem 1 and Lemmas 1, 2 and 3 follow from Figure 1.

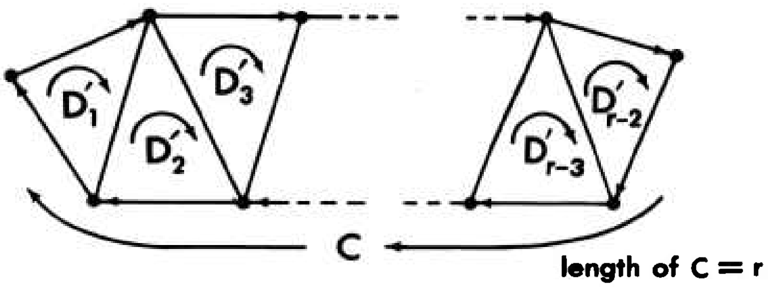

Figure 2.

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